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The resonance effect of a disk oscillating about a state of steady rotation

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The rotating system of an infinite disk beneath an unbounded fluid can exhibit resonance if the disk performs torsional oscillations at a certain frequency. This effect is examined in detail, and the solution is shown to depend crucially upon the existence of a small, steady departure from the basic rotational state in the far field.

1. Introduction

The boundary-layer flow of a tidal or gravity wave, which travels over a smooth sea bed, was examined by Hunt & Johns (1963). A solution was obtained by a linearization procedure, which essentially proposed a first-order balance between the viscous, accelerative and Coriolis forces in the equation of motion. An interesting feature of the analysis was that when the latitude took a certain value, which depended on the wave frequency and the strength of the earth's rotation, their linear solution was no longer valid. They did not examine this singularity any further. Benney (1965) considered a problem, which contained all the essential features of Hunt & Johns's problem, but with a simpler geometry. In this paper, we shall be examining this second problem and, in particular, we shall be concerned with analyzing the singularity it contains. This singularity is a resonance phenomenon, and is directly analogous to the effect found by Hunt & Johns.

A semi-infinite fluid of density ρ and kinematic viscosity ν lies above an infinite disk. The disk is assumed to have a uniform angular velocity Ω, upon which are superimposed small, torsional oscillations of magnitude ω, frequency σ. The oscillations are small in the sense that $\omega \ll \Omega$. Far from the disk we also suppose the fluid to have a constant speed of rotation. Although we will be primarily interested in the effect of an oscillating boundary on a single, basic, rotational state, we shall not immediately equate the angular velocity of the fluid at infinity to Ω. We shall assume there is a small difference of $O(\omega)$ or less. This is because (as shown in Jones 1969) a zero-perturbation, far-field velocity does not necessarily correspond to a distant boundary that rotates with a zero perturbation. The difference in the steady component of the azimuthal velocities can be supported by the formation of Ekman layers at both near and far boundaries. This should be compared with the time-dependent component of the flow where only a modified Stokes layer near the oscillating (shearing) disk can form.

Thus, we generalize the infinite fluid problem by allowing a steady rotational state in the far field with an arbitrarily assigned value. There is a tacit assumption in this, that to any value chosen for the far-field velocity there corresponds a two-boundary problem, which is defined by suitable boundary conditions on the far surface. Thus, while our problem is quite generally posed, we are working under limitations. Given a one-boundary problem, we cannot necessarily determine the two-boundary problem to which it corresponds (assuming it exists). Nor can we necessarily go the other way. Nevertheless, this approach does enable us to analyze the forms possible for the solution of the more realistic, two-boundary problem.

The resonance point, our main concern, occurs when $\sigma = 2\Omega$. The solutions that we obtain in this case all depend on the existence of the far-field perturbation velocity. They all have double boundary-layer structures, and are all determined by the method of matched asymptotic expansions. However, the scaling and exact form of the outer layer in each case depend critically on the order of magnitude of the perturbation at infinity. The three orders of magnitude considered are: $O(\omega)$, $O(\omega^2/\Omega)$ and $O(\omega^4/\Omega^3)$. For far field perturbations smaller than this, we have not managed to obtain any solution, although the possible forms that a solution may take are discussed. Finally, the particular two-boundary problem is briefly examined, where a distant disk with an angular velocity Ω bounds the fluid. This is shown to correspond to an interior perturbation velocity of $O(\omega^2/\Omega)$.

2. Equations

Cylindrical co-ordinates $(\tilde{r}, \tilde{\theta}, \tilde{z})$ are chosen with accompanying velocity components $(\tilde{u}, \tilde{v}, \tilde{w})$. The disk is defined by the equation $\tilde{z} = 0$, with $\tilde{r} = 0$ as its axis of rotation. Assuming axial symmetry, the equations of motion are:

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{r}} + \tilde{w}\frac{\partial \tilde{u}}{\partial \tilde{z}} - \frac{\tilde{v}^2}{\tilde{r}} = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial \tilde{r}} + \nu\left(\nabla^2\tilde{u} - \frac{\tilde{u}}{\tilde{r}^2}\right), \tag{2.1}$$

$$\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{r}} + \tilde{w}\frac{\partial \tilde{v}}{\partial \tilde{z}} + \frac{\tilde{u}\tilde{v}}{\tilde{r}} = \nu\left(\nabla^2\tilde{v} - \frac{\tilde{v}}{\tilde{r}^2}\right), \tag{2.2}$$

$$\frac{\partial \tilde{w}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{w}}{\partial \tilde{r}} + \tilde{w}\frac{\partial \tilde{w}}{\partial \tilde{z}} = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial \tilde{z}} + \nu\nabla^2\tilde{w}, \tag{2.3}$$

$$\frac{\partial \tilde{u}}{\partial \tilde{r}} + \frac{\tilde{u}}{\tilde{r}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0, \tag{2.4}$$

where $\tilde{t}$ is time and $\tilde{P}$ pressure. The boundary conditions are:

$$\left.\begin{aligned} \tilde{u} = \tilde{w} = 0, \\ \tilde{v} = \Omega\tilde{r} + 2\omega\tilde{r}\,e^{i\sigma\tilde{t}}, \end{aligned}\right\} \quad \text{on} \quad \tilde{z} = 0, \quad \text{all} \quad \tilde{r}, \tilde{t}, \tag{2.5a}$$

and
$$\left.\begin{aligned}\tilde{u} &\to 0,\\ \tilde{v} &\to \Omega\tilde{r}+A\omega\tilde{r},\end{aligned}\right\}\quad \text{as}\quad \tilde{z}\to\infty,\quad \text{all}\quad \tilde{r},\,\tilde{t}, \tag{2.5b}$$

where A is a constant. In using complex notation, it is to be understood that the real part of the equation is to be taken. The factor of two in the second boundary condition is a matter of convenience.

Several simplifications of these equations are possible. (i) From the geometry of the problem, we assume the radial variable $\tilde{r}$ can be eliminated as a similarity variable. (ii) The equation of continuity allows the introduction of a Stokes stream function. (iii) It is convenient to change to a rotating co-ordinate system. (iv) We shall introduce dimensionless variables using as our length scale the Ekman layer thickness $(\nu/\Omega)^{\frac{1}{2}}$. Thus, we define:

$$\left.\begin{aligned}\tilde{u} &= 2\omega\tilde{r}\frac{\partial F}{\partial z}(z,t), & \tilde{w} &= -2\omega\,(2\nu/\Omega)^{\frac{1}{2}}F(z,t),\\ \tilde{v} &= \Omega\tilde{r}+2\omega\tilde{r}G(z,t), & \tilde{z} &= (\nu/2\Omega)^{\frac{1}{2}}z,\quad \tilde{t}=\sigma^{-1}t.\end{aligned}\right\} \tag{2.6}$$

The most general form for the radial pressure gradient, compatible with the assumption of radial similarity, and with the boundary conditions at infinity, is

$$\frac{1}{\rho}\frac{\partial\tilde{P}}{\partial\tilde{r}} = \Omega^2\tilde{r}+4\omega\Omega K\tilde{r}, \tag{2.7}$$

where
$$K = \frac{1}{4}\left(2A+A^2\frac{\omega}{\Omega}\right).$$

Under these transformations, (2.1) and (2.2) become

$$\left.\begin{aligned}F'''-p\frac{\partial F'}{\partial t}+G &= K+\epsilon[F'^2-2F\,F''-G^2],\\ G''-p\frac{\partial G}{\partial t}-F' &= 2\epsilon[GF'-FG'],\end{aligned}\right\} \tag{2.8}$$

where ϵ and p are defined by

$$p = \sigma/2\Omega,\quad \epsilon = \omega/\Omega \ll 1, \tag{2.9}$$

and a dash represents a differentiation with respect to z.

Equation (2.3) serves only to determine the axial pressure gradient while the equation of continuity is automatically satisfied. The transformed boundary conditions are:

$$F = \frac{\partial F}{\partial z} = 0,\quad G = e^{it}\quad \text{on}\quad z=0, \tag{2.10a}$$

$$\frac{\partial F}{\partial z}\to 0,\quad G\to\{1-(1+4\epsilon K)^{\frac{1}{2}}\}/2\epsilon\quad \text{as}\quad z\to\infty, \tag{2.10b}$$

and we shall be searching for purely periodic solutions.

3. Description of resonance

We shall look for an asymptotic solution as $\epsilon\to 0$. The straightforward approach to this is to assume expansions for F and G which are power series in ϵ. Because we are looking for a purely periodic solution, however, the boundary conditions suggest a more useful expansion is available to us, namely,

$$F \sim F_{00}(z)+F_{01}(z)\,e^{it}+\epsilon[F_{10}(z)+F_{11}(z)\,e^{it}+F_{12}(z)\,e^{2it}]+O(\epsilon^2), \tag{3.1}$$

and similarly for G.

These are substituted into (2.8) to (2.10). The equations for F_{00} and G_{00} are obtained by equating the steady terms whose coefficients are ϵ to the power zero.

$$\left.\begin{aligned} F'''_{00}+G_{00} &= K,\\ G''_{00}-F'_{00} &= 0. \end{aligned}\right\} \tag{3.2a}$$

The boundary conditions are:

$$\left.\begin{aligned} &F_{00} = F'_{00} = G_{00} = 0, \quad \text{on} \quad z = 0,\\ &F'_{00}\to 0, \quad G_{00}\to K, \quad \text{as} \quad z\to\infty; \end{aligned}\right\} \tag{3.2b}$$

and the solution is:

$$\left.\begin{aligned} F_{00} &= \frac{K}{2}\left[\frac{1+i}{\sqrt{2}}\exp\left(-\frac{1+i}{\sqrt{2}}z\right)+\frac{1-i}{\sqrt{2}}\exp\left(-\frac{1-i}{\sqrt{2}}z\right)-\sqrt{2}\right],\\ G_{00} &= \frac{K}{2}\left[2-\exp\left(-\frac{1+i}{\sqrt{2}}z\right)-\exp\left(-\frac{1-i}{\sqrt{2}}z\right)\right]. \end{aligned}\right\} \tag{3.3}$$

The equations for F_{01} and G_{01} are derived similarly. They are:

$$F'''_{01}-ipF'_{01}+G_{01} = 0, \tag{3.4a}$$

$$G''_{01}-ipG_{01}-F'_{01} = 0, \tag{3.4b}$$

with the boundary conditions

$$F_{01} = F'_{01} = 0, \quad G_{01} = 1, \quad z = 0, \tag{3.5}$$

$$F'_{01}\to 0, \quad G_{01}\to 0, \quad z\to\infty. \tag{3.6}$$

When $p \neq 1$, the solution of these equations is:

$$\left.\begin{aligned} F_{01} &= \frac{i}{2}\left[\frac{1}{\alpha}(e^{\alpha z}-1)+\frac{1}{\beta}(1-e^{\beta z})\right],\\ G_{01} &= \tfrac{1}{2}[e^{\alpha z}+e^{\beta z}], \end{aligned}\right\} \tag{3.7}$$

where $\left.\begin{aligned} \alpha &= [i(p+1)]^{\frac{1}{2}}\\ \beta &= [i(p-1)]^{\frac{1}{2}} \end{aligned}\right\}$ (negative real part).

We now investigate the solution as $p\to 1$. The most general solution of (3.4) (which satisfies the boundary conditions on the disk, (3.5)) is

$$\left.\begin{aligned} F_{01} &= -\frac{1+i}{4}e^{-(1+i)z}+\frac{1+i}{4}-\frac{i}{2}z+bz^2,\\ G_{01} &= \tfrac{1}{2}[1+e^{-(1+i)z}]+2ibz, \end{aligned}\right\} \tag{3.8}$$

where b is a constant. But these expressions are unable to satisfy the boundary conditions at infinity (3.6). To see why this happens, we return to (3.4), and consider the balance of terms that exist far from the disk. At a sufficient distance, the highest differentiated term (the viscous term) declines in importance, and can be neglected to first order in comparison with the others. Thus the first-order equations are:

$$\left.\begin{aligned} -piF'_{01}+G_{01} &= 0, \\ -piG_{01}-F'_{01} &= 0. \end{aligned}\right\} \quad (3.9)$$

Provided $p \neq 1$, these have a solution which correctly describes the form of the far field solution. When $p = 1$, however, the equations are degenerate and become identical. This represents a resonance or feedback situation between the accelerative and Coriolis forces. Instead of determining the flow, (3.9) now only provides a first-order restriction. A supplementary equation for the flow is needed, and can be obtained by differentiating (3.4*b*) with respect to time, and subtracting it from (3.4*a*). The result of this is

$$F''''_{01}-iG''_{01} = 0. \quad (3.10)$$

Thus, although the flow has to satisfy a first-order restriction involving the Coriolis and accelerative forces, its actual form is determined by viscous action.

Unfortunately, the form of solution allowed by (3.10), a second degree polynomial in z, cannot be matched to the inner solution (3.8) near the disk and also remain bounded at infinity. In fact, (3.10) is not sufficiently exact. Our determining equation must be derived from the complete equations (2.8) (with $p = 1$), because a solution is only possible if a far-field balance can be achieved between the viscous terms (which we have assumed small, in some sense) and the hitherto neglected non-linear terms. Proceeding as above, we differentiate the second of equations (2.8) with respect to time, and subtract from the first

$$\left\{F'''' - iG'' = \epsilon\left[F'^2 - 2FF' - G^2 - 2\frac{\partial}{\partial t}(GF' - FG')\right]\right\}e^{it} \text{ terms only.} \quad (3.11)$$

We shall refer to this henceforth as the complementary equation, since it determines the solution when used in conjunction with the first-order restriction (3.10).

4. Resonance when $K = O(1)$

We shall now proceed more formally. From the previous heuristic discussion, we expect to find inner and outer solutions for the flow, which can be matched together. The inner solution is clearly (3.8), and we shall assume that $b = 0$ in these two equations ($b \neq 0$ leads to homogeneous boundary conditions for the outer solution, which are unlikely to yield a solution). For the outer variables we define:

$$z = \epsilon^{\frac{1}{2}}\eta, \quad f = \epsilon^{\frac{1}{2}}F, \quad g = G, \quad (4.1)$$

which are suggested by (3.11) and (3.8). The original (2.8) thus becomes:

$$\left.\begin{aligned} \epsilon\frac{\partial^3 f}{\partial\eta^3} - \frac{\partial^2 f}{\partial\eta\,\partial t} + g &= K + \epsilon\left[\left(\frac{\partial f}{\partial\eta}\right)^2 - 2f\frac{\partial^2 f}{\partial\eta^2} - g^2\right], \\ \epsilon\frac{\partial^2 g}{\partial\eta^2} - \frac{\partial g}{\partial t} - \frac{\partial f}{\partial\eta} &= 2\epsilon\left[g\frac{\partial f}{\partial\eta} - f\frac{\partial g}{\partial\eta}\right]. \end{aligned}\right\} \quad (4.2)$$

We assume expansions for the outer variables of the form

$$f \sim f_{00}(\eta) + f_{01}(\eta)\, e^{it} + O(\epsilon^{\frac{1}{2}}) \tag{4.3}$$

and similarly for g. These are substituted into (4.2) and coefficients of different powers of ϵ are equated. The steady zero-order terms lead to the result,

$$f_{00} = 0, \quad g_{00} = K, \tag{4.4}$$

after matching to the steady inner solution (3.3).

The first harmonic, zero-order terms of both equations are:

$$g_{01} = i\frac{\partial f_{01}}{\partial \eta}, \tag{4.5}$$

and the complementary equation has to be derived as before by differentiating the second of equations (4.2) with respect to time, and subtracting from the first. If (4.4) are then used, the resulting expression simplifies to

$$\frac{\partial^3 f_{01}}{\partial \eta^3} - i\frac{\partial^2 g_{01}}{\partial \eta^2} = -2Kg_{01} - 2Ki\frac{\partial f_{01}}{\partial \eta}. \tag{4.6}$$

Finally, the solution of these equations, which is bounded at infinity and matches to (3.8), is:

$$\left.\begin{aligned} f_{01} &= \frac{1-i}{4\sqrt{K}}(1 - e^{-(1-i)\sqrt{K}\eta}), \\ g_{01} &= \tfrac{1}{2}\, e^{-(1-i)\sqrt{K}\eta}. \end{aligned}\right\} \tag{4.7}$$

It should be pointed out before leaving § 5 that the proposed inner expansion (3.1) is now incorrect. The matching process induces additional terms in the inner solution, whose orders are half-powers of ϵ, and the expansion (3.1) must be amended to allow for this. Equations (3.3) and (3.8) are still the first-order solutions, however.

5. Resonance when $K = O(\epsilon)$

The solution of § 4 is invalid when $K = O(\epsilon)$, because there is no longer any first-order steady motion. Consequently, a balance cannot be achieved in the complementary equation (3.11), which involves the first products in the non-linear terms, since the first products are either steady or second harmonic. To obtain the necessary simple harmonic terms, one must consider second-product terms of the right-hand side.

We write

$$K = \epsilon\mathscr{K}, \tag{5.1}$$

and (2.8) becomes

$$\left.\begin{aligned} F'''' - \frac{\partial F'}{\partial t} + G &= \epsilon[\mathscr{K} + F'^2 - 2FF'' - G^2], \\ G'' - \frac{\partial G}{\partial t} - F' &= 2\epsilon[GF' - FG'], \end{aligned}\right\} \tag{5.2}$$

with boundary conditions,

$$\left.\begin{aligned} F = F' = 0, \quad G = e^{it}, \quad \text{on} \quad z = 0, \\ F' \to 0, \quad G + \epsilon G^2 \to \epsilon \mathscr{K}, \quad \text{as} \quad z \to \infty. \end{aligned}\right\} \tag{5.3}$$

We assume an inner expansion for F of

$$F \sim F_{01}(z)\, e^{it} + \epsilon[F_{10}(z) + F_{11}(z)\, e^{it} + F_{12}(z)\, e^{2it}] + O(\epsilon^2), \tag{5.4}$$

and a similar one for G.

As usual, these are substituted into the equations, and terms which are multiplied by the same power of ϵ and by the same harmonic are equated. The boundary conditions are obtained similarly except, again (since this is only an inner solution) the outer boundary conditions of (5.3) are relaxed, and instead we merely forbid the occurrence of exponentially growing terms in the solution. The first-order, simple harmonic terms have the same equations and boundary conditions as before, and thus have the same solution (3.8). The only other inner solution we shall need is the second-order, steady component of the motion. The equations are

$$\left.\begin{aligned} F'''_{10} + G_{10} &= \mathscr{K} + \tfrac{1}{2}F'_{01}\bar{F}'_{01} - F_{01}\bar{F}''_{01} - \tfrac{1}{2}G_{01}\bar{G}_{01}, \\ G''_{10} - F'_{00} &= G_{01}\bar{F}'_{01} - F_{01}\bar{G}'_{01}, \end{aligned}\right\} \tag{5.5}$$

where a bar denotes a complex conjugate. The solution takes the form

$$F_{10} = \tfrac{1}{4}[-ize^{-(1+i)z} - (4+3i)\, e^{-(1+i)z} - \tfrac{1}{34}e^{-2z}] + a\, e^{-(1+i)z/\sqrt{2}} + b\, e^{-(1-i)z/\sqrt{2}} + c, \tag{5.6a}$$

$$G_{10} = \mathscr{K} + \tfrac{1}{4}[(1+i)z\, e^{-(1+i)z} + (6-i)\, e^{-(1+i)z} - \tfrac{8}{34}e^{-2z}] - \frac{a(1-i)}{\sqrt{2}}\, e^{-(1+i)z/\sqrt{2}} - \frac{b(1+i)}{\sqrt{2}}\, e^{-(1-i)z/\sqrt{2}}, \tag{5.6b}$$

where a, b, c are constants whose actual values we shall not need.

For the outer expansion, the use of the new scaled variables,

$$\eta = \epsilon z, \quad f = \epsilon F, \quad g = G, \tag{5.7}$$

is suggested by (3.8) and (3.11). Series expansions for f and g are then assumed with the same form as the inner expansion, i.e.

$$f \sim f_{01}(\eta)\, e^{it} + \epsilon[f_{10}(\eta) + f_{11}(\eta)\, e^{it} + f_{12}(\eta)\, e^{2it}] + \epsilon^2[f_{20}(\eta) + f_{21}(\eta)\, e^{it} + f_{22}(\eta)\, e^{2it} + f_{23}(\eta)\, e^{3it}] + O(\epsilon^3). \tag{5.8}$$

The usual procedure is followed. The new variables (5.7), and their expansions (5.8), are substituted into the equations of motion (5.2), and terms which are multiplied by the same time-harmonic and the same power of ϵ are equated. The following set of equations are obtained: first order, first harmonic:

$$g_{01} = if'_{01}; \tag{5.9}$$

second order, steady:

$$g_{10} + \bar{g}_{10} = 2\mathscr{K} + f'_{01}\bar{f}'_{01} - f_{01}\bar{f}''_{01} - f''_{01}\bar{f}_{01} - g_{01}\bar{g}_{01}, \tag{5.10a}$$

$$-(f_{10} + \bar{f}_{10})' = g_{01}\bar{f}'_{01} + \bar{g}_{01}f'_{01} - f_{01}\bar{g}'_{01} - \bar{f}_{01}g'_{01}; \tag{5.10b}$$

second order, second harmonic:

$$-2if'_{12}+g_{12} = \tfrac{1}{2}(f'_{01}f'_{01}-2f_{01}f''_{01}-g_{01}g_{01}), \tag{5.11a}$$

$$2ig_{12}+f'_{12} = (f_{01}g'_{01}-g_{01}f'_{01}); \tag{5.11b}$$

third order, first harmonic:

$$\begin{aligned} f'''_{01}-if'_{21}+g_{21} &= (f_{10}+\bar{f}_{10})'f'_{01}+f'_{12}\bar{f}'_{01}-(f_{10}+\bar{f}_{10})f''_{01}-(f_{10}+\bar{f}_{10})''f_{01} \\ &\quad -f_{12}\bar{f}''_{01}-f''_{12}\bar{f}_{01}-(g_{10}+\bar{g}_{10})g_{01}-g_{12}\bar{g}_{01}, \end{aligned} \tag{5.12a}$$

$$\begin{aligned} g''_{01}-ig_{21}-f'_{21} &= (g_{10}+\bar{g}_{10})f'_{01}+g_{01}(f_{10}+\bar{f}_{10})'+g_{12}\bar{f}'_{01}+\bar{g}_{01}f'_{12} \\ &\quad -(f_{10}+\bar{f}_{10})g'_{01}-f_{01}(g_{10}+\bar{g}_{10})'-f_{12}\bar{g}'_{01}-\bar{f}_{01}g'_{12}. \end{aligned} \tag{5.12b}$$

Other equations, such as those for f_{11} and g_{11}, will not be needed. The dash now represents a differentiation with respect to the new variable η, of course.

The boundary conditions for the equations are matching to the inner solution (3.8) and (5.6):

$$\left.\begin{aligned} f_{01}(0) = 0, \quad f'_{01}(0) = -\tfrac{1}{2}i, \\ f_{10}(0) = 0, \end{aligned}\right\} \tag{5.13a}$$

and the outer restrictions of (5.3), i.e.

$$f'_{01}(\infty) = f'_{10}(\infty) = 0, \quad g_{10}(\infty) = \mathscr{K}. \tag{5.13b}$$

Our final objective is to obtain an equation for f_{01}. Two preliminary steps are necessary. First, equation (5.10b) has to be integrated to give an expression for $(f_{10}+\bar{f}_{10})$. After (5.9) has been used to eliminate g_{01}, this can be done without difficulty

$$f_{10}+\bar{f}_{10} = i(f'_{01}\bar{f}_{01}-f_{01}\bar{f}'_{01}), \tag{5.14}$$

and the constant of integration is zero, because of the boundary conditions (5.13a). Secondly, (5.11) must be rearranged to give separate expressions for f'_{12} and g_{12}. The equations can then be reduced to a single equation for f_{01} by the following procedure:

(i) Equation (5.12b) is multiplied by i, and subtracted from (5.12a), in order to eliminate both f_{21} and g_{21}. The resulting equation, of course, corresponds to the complementary equation (3.1).

(ii) From this equation, f_{10}, g_{10}, f_{12} and g_{12} are eliminated by substitution from (5.10), (5.11) and (5.14). It is interesting that f_{12} does not appear explicitly in the complementary equation, but appears only in a differentiated form, f'_{12} or f''_{12}. This is fortunate, since it is impossible to integrate f'_{12} to give a closed form expression for f_{12}, as has been done for f_{10}.

(iii) From the equation produced by step (ii), g_{01} may be eliminated by use of (5.9). It is then discovered that all the non-linear products cancel identically, thus leaving the equation,

$$f'''_{01}+2i\mathscr{K}f'_{01} = 0.$$

The solution of this satisfying the boundary conditions is

$$f_{01} = \frac{1-i}{4\sqrt{\mathscr{K}}}[1-e^{-(1-i)\sqrt{\mathscr{K}}\eta}].$$

Allowing for the scaling difference, this result is the same as the previous result (4.7) for the outer boundary layer. The reason is that the equations of motion are doubly degenerate. The original equations are degenerate, because a combination of them is possible that eliminates the first-order harmonic terms. But the complementary equation so produced is also degenerate, since the second-products of the non-linear terms do not have any simple harmonic part, although one would expect one. The exception is the interaction between the first-order oscillatory flow, and the far-field, steady, azimuthal velocity. But the balance achieved with this is of course the same as before, and hence the results are equivalent.

6. Resonance when $K = O(\epsilon^3)$

The solution of the previous section is invalid when $K = O(\epsilon^3)$ (i.e. $\mathscr{K} = O(\epsilon^2)$). The neglected fourth-products of the non-linear terms then have a harmonic part which is comparable to the terms that are retained to reach a solution.

To investigate this possibility, we write:

$$\mathscr{K} = \epsilon^2 k, \quad (K = \epsilon^3 k), \tag{6.1}$$

and substitute this into (5.2). The initial analysis proceeds much as before. Inner expansions may be chosen of the same form as (5.4). These lead to the usual first-order, harmonic solution (3.8), and to the same second-order steady solution (5.6), but with the modification that $\mathscr{K}$ no longer appears on the right-hand side of (5.6*b*). The boundary conditions then give us

$$\left.\begin{aligned} a &= \frac{5(47+33i)}{136\sqrt{2}}, \\ b &= \frac{-3+5i}{136\sqrt{2}}, \\ c &= \tfrac{1}{136}(137-116\sqrt{2}), \end{aligned}\right\} \tag{6.2}$$

and unlike the previous case, the numerical value of c turns out to be relevant to the solution.

For the outer expansion, the appropriately scaled variables are:

$$\eta = \epsilon^2 z, \quad f = \epsilon^2 F, \quad g = G. \tag{6.3}$$

However, it is no longer possible to utilize expansions for f and g of the form (5.8). For suppose one did so, and continued as before. The zero-order terms would give only a single equation connecting f_{01} and g_{01}, and similarly the second-order terms would provide only a single relation between f_{21} and g_{21} (the odd-powers harmonics, such as f_{11}, would be zero). So after eliminating f_{41} and g_{41} from the fourth-order equations, the resulting equation could be reduced to give only an equation for f_{01}, if f_{21} and g_{21} chanced to appear in it in the combined form of $(f_{21} - ig_{21})$. Since the equations are non-linear, there is no guarantee that this will happen. Clearly, however, this fault only arises because of the form of the perturbation expansion. The difficulty may be avoided by assuming a leading term expansion. This is essentially a Fourier series since any harmonic e^{int} appears only once; also, the coefficients are scaled so that their expected magnitudes are $O(1)$. Thus,

$$f \sim f_1(\eta;\epsilon)\,e^{it} + \epsilon(f_0(\eta;\epsilon) + f_2(\eta;\epsilon)\,e^{2it}) + \epsilon^2 f_3(\eta;\epsilon)\,e^{3it} + \epsilon^3 f_4(\eta;\epsilon)\,e^{4it} + O(\epsilon^4), \quad (6.4)$$

and similarly for g.

The outer scalings (6.3) and the new expansions are substituted into the equations of motion (5.2). It is no longer possible to equate coefficients of different powers of ϵ, of course. Instead the equations are Fourier analyzed. Thus one obtains:

first harmonic terms:

$$\begin{aligned}\epsilon^4 f_1'''' - i f_1' + g_1 \sim \epsilon^2[&(f_0+\bar{f}_0)' f_1' + f_2'\bar{f}_1' - (f_0+\bar{f}_0) f_1'' - (f_0+\bar{f}_0)'' f_1 - f_2 \bar{f}_1'' - \bar{f}_1 f_2'' \\ &- (g_0+\bar{g}_0) g_1 - g_2\bar{g}_1] + \epsilon^4[\bar{f}_2' f_3' - \bar{f}_2 f_3'' - f_3 \bar{f}_2'' - \bar{g}_2 g_3] + O(\epsilon^6),\end{aligned} \quad (6.5a)$$

$$\begin{aligned}\epsilon^4 g_1'' - i g_1 - f_1' \sim \epsilon^2[&(g_0+\bar{g}_0) f_1' + g_1(f_0+\bar{f}_0)' + g_2\bar{f}_1' + \bar{g}_1 f_2' - (f_0+\bar{f}_0) g_1' - f_1(g_0+\bar{g}_0)' \\ &- f_2\bar{g}_1' - \bar{f}_1 g_2'] + \epsilon^4[\bar{g}_2 f_3' + g_3\bar{f}_2' - \bar{f}_2 g_3' - f_3\bar{g}_2'] + O(\epsilon^6);\end{aligned} \quad (6.5b)$$

steady terms:

$$\begin{aligned}(g_0+\bar{g}_0) \sim 2\epsilon^2 k + [&f_1'\bar{f}_1' - f_1\bar{f}_1'' - \bar{f}_1 f_1'' - g_1\bar{g}_1] + \epsilon^2[\tfrac{1}{2}(f_0+\bar{f}_0)'(f_0+\bar{f}_0)' + f_2'\bar{f}_2' \\ &- (f_0+\bar{f}_0)(f_0+\bar{f}_0)'' - f_2\bar{f}_2'' - \bar{f}_2 f_2'' - \tfrac{1}{2}(g_0+\bar{g}_0)(g_0+\bar{g}_0) - g_2\bar{g}_2] + O(\epsilon^4);\end{aligned} \quad (6.6a)$$

$$\begin{aligned}(f_0+\bar{f}_0)' \sim [&f_1\bar{g}_1' + \bar{f}_1 g_1' - g_1\bar{f}_1' - \bar{g}_1 f_1'] + \epsilon^2[(f_0+\bar{f}_0)(g_0+\bar{g}_0)' + f_2\bar{g}_2' + \bar{f}_2 g_2' \\ &- (g_0+\bar{g}_0)(f_0+\bar{f}_0)' - g_2\bar{f}_2' - \bar{g}_2 f_2'] + O(\epsilon^4);\end{aligned} \quad (6.6b)$$

second harmonic terms:

$$\begin{aligned}-2i f_2' + g_2 \sim [&\tfrac{1}{2} f_1' f_1' - f_1 f_1'' - \tfrac{1}{2} g_1 g_1] + \epsilon^2[f_3'\bar{f}_1' + (f_0+\bar{f}_0)' f_2' - f_3\bar{f}_1'' - \bar{f}_1 f_3'' \\ &- (f_0+\bar{f}_0) f_2'' - f_2(f_0+\bar{f}_0)'' - g_3\bar{g}_1 - (g_0+\bar{g}_0) g_2] + O(\epsilon^4),\end{aligned} \quad (6.7a)$$

$$\begin{aligned}2i g_2 + f_2' \sim [&f_1 g_1' - g_1 f_1'] + \epsilon^2[f_3\bar{g}_1' + \bar{f}_1 g_3' + (f_0+\bar{f}_0) g_2' + f_2(g_0+\bar{g}_0)' \\ &- g_3\bar{f}_1' - \bar{g}_1 f_3' - (g_0+\bar{g}_0) f_2' - g_2(f_0+\bar{f}_0)'] + O(\epsilon^4);\end{aligned} \quad (6.7b)$$

third harmonic terms:

$$-3i f_3' + g_3 \sim [f_1' f_2' - f_1 f_2'' - f_2 f_1'' - g_1 g_2] + O(\epsilon^2), \quad (6.8a)$$

$$3i g_3 + f_3' \sim [f_1 g_2' + f_2 g_1' - g_1 f_2' - g_2 f_1'] + O(\epsilon^2). \quad (6.8b)$$

The boundary conditions are:

$$\frac{\partial f_i}{\partial \eta} \to 0, \quad \text{as} \quad \eta \to \infty, \quad (6.9a)$$

and from matching to the inner solution

$$\left.\begin{aligned} f_1 &\sim -\tfrac{1}{2} i\eta + \epsilon^2(\tfrac{1}{4}(1+i)) \\ f_0 &\sim \epsilon^2 c \end{aligned}\right\}, \quad \text{as} \quad \eta \to 0, \quad (6.9b)$$

where the value of c is given in (6.2).

To obtain the final solution, these equations are treated in a very similar fashion to the equations of §5. Equation (6.5*b*) is multiplied by i, and subtracted from (6.5*a*), to leave the complementary equation. The rest is then substitution and re-substitution into the right-hand side of this equation, to reduce it to terms that are $O(\epsilon^4)$, and which are functions of f_1 only. It should be realized that, when doing this, (6.5)–(6.8) can be used in many different forms. For instance, (6.5a) may be truncated to:

$$g_1 \sim if'_1 + O(\epsilon^2). \tag{6.10}$$

Provided this is substituted only into terms that have already been multiplied by ϵ^4, the omitted terms will be $O(\epsilon^6)$; and these terms are in any case neglected in the final equation.

In the course of the analysis, it is necessary to integrate (6.6b) again to first order:

$$f_0 + \bar{f}_0 \sim i(\bar{f}_1 f'_1 - f_1 \bar{f}'_1) + O(\epsilon^2);$$

but, once more, it is never necessary to integrate any expression for f_2, since f_2 appears only in a differentiated form.

The most interesting stage in the reduction is reached with the equation,

$$\begin{aligned}\epsilon^4(f'''_1 - ig''_1) = \epsilon^2[(g_1 + if'_1)'(\bar{f}_1 f'_1 + i(f_0 + \bar{f}_0) - if_1\bar{g}_1)] - 2ikf'_1,\\ + 2i\epsilon^4\{(00\bar{0}2\bar{3}) + (01\bar{1}\bar{1}2) + (\bar{0}\bar{0}122) + (0\bar{0}\bar{1}22),\\ - (\bar{0}11\bar{1}2) - (00\bar{1}2\bar{2})\} + \epsilon^4(022)\bar{f}_2 + O(\epsilon^6),\end{aligned} \tag{6.11}$$

where a convenient shorthand is used to represent the fifth (or third) product of f_1: the symbol f_1 is omitted, the number refers to the derivative, and the bar to its complex conjugacy. To evaluate the square bracket, it is necessary to integrate $(f_0 + \bar{f}_0)'$ to second order. To do this, the bracket as a whole is transformed into an integral,

$$\begin{aligned}[\ldots] = \epsilon^2(g_1 + if'_1)'\{[\bar{f}_1 f'_1 + i(f_0 + \bar{f}_0) - if_1\bar{g}_1]_{\eta=0}\\ + \int_0^\eta [\bar{f}'_1 f'_1 + \bar{f}_1 f''_1 - if'_1\bar{g}_1 - if_1\bar{g}'_1 + i(f_0 + \bar{f}_0)']\,d\eta^*\};\end{aligned} \tag{6.12}$$

and it is then possible to substitute for everything that appears under the integral sign. All terms of $O(\epsilon^2)$ eventually vanish, and the remaining $O(\epsilon^4)$ terms can be evaluated explicitly by integration by parts. It turns out that the terms thus obtained cancel identically with the non-linear terms of (6.11), and so leave the linear equation,

$$f'''_1 - \theta f''_1 + 2ikf'_1 \sim O(\epsilon^2), \tag{6.13}$$

where θ is defined by

$$\theta = \frac{1}{\epsilon^2}[i\bar{f}_1 f'_1 - (f_0 + \bar{f}_0) + f_1\bar{g}_1]_{\eta=0},$$

$$\theta \sim \tfrac{1}{4} + \left(\frac{116\sqrt{2} - 137}{68}\right) + O(\epsilon^2). \tag{6.14}$$

The solution that satisfies the boundary conditions is thus

$$f_1 \sim -\frac{i}{2[\theta - (\theta^2 - ik)^{\frac{1}{2}}]}(e^{[\theta - (\theta^2 - 8ik)^{\frac{1}{2}}\,\eta)}). \tag{6.15}$$

The constant θ has a physical significance, since it represents the steady outflow velocity of the fluid to first order at infinity (i.e. $f_0 + \bar{f}_0$ at $\eta = \infty$). This is easy to see by substituting the boundary conditions (6.9a) into (6.11), and comparing the result with (6.13). The tragedy of θ is that $\theta > 0$, and we have an outflow, not an inflow, at infinity. The result of this is that, if we choose k to have an even lower magnitude than $O(\epsilon^3)$, the outer boundary layer given by (6.15) is unstable. As $k \to 0$, the outer boundary-layer thickness becomes infinite, because the convective term alone cannot balance the viscous force, but only serve to intensify it. No further analysis is attempted for this case although several possibilities suggest themselves:

(i) The analysis could be continued. If an even larger scaling length is assumed in the complementary equation, the viscous terms would become negligible. As the overall magnitude of $\theta f_1''$ would also be reduced, however, it is feasible that some balance in the complementary equation is possible, where the balancing terms are drawn solely from the non-linear right-hand side of (3.11).

(ii) Mathematical considerations. One of several assumptions that have been made could be incorrect. Principally we have assumed (*a*) a similarity solution in $\tilde{r}$, (*b*) a purely periodic solution, (*c*) an asymptotic solution in ϵ, (*d*) that the method of matched asymptotic expansions is valid, (*e*) that $b = 0$ in the inner solution (3.8).

(iii) The intrusion of reality. The flow may be turbulent. Or a solution may exist only in a bounded fluid (either radially or azimuthally). Conversely, it is possible that $k = 0$ could never actually be realized.

7. The axially bounded case

Finally, we shall investigate briefly what resonance form occurs, if a second disk with no perturbation velocity is present in the fluid at a large distance from the first one. We shall actually assume that resonance is of the second type ($\mathscr{K} = O(1)$), and simply confirm that the scalings of §5 lead to a consistent, non-zero value for $\mathscr{K}$. To do this, we shall suppose the fluid can be divided into four regions, namely, an inner boundary layer near the first disk (really a modified Stokes layer), an outer boundary layer, then a large intermediate region, through which the steady velocities remain principally unchanged, and finally an Ekman layer at the second disk. It is assumed that the intermediate region is sufficiently large that the harmonic velocities (which die slowly away in a bounded domain) are negligible at the far disk.

Inner boundary layer

The solution is described in §5. The steady solution has the form (5.6). To satisfy the homogeneous boundary conditions,

$$F_{10} = F'_{10} = G_{10} = 0, \tag{7.1}$$

it follows that

$$2c + \mathscr{K} = \frac{137\sqrt{2} - 232}{136}. \tag{7.2}$$

Outer boundary layer

The additional equations

$$g_{11} = if'_{11}, \tag{7.3}$$

$$(f_{20} + \bar{f}_{20})' = f_{01}\bar{g}'_{11} + \bar{f}_{01}g'_{11} + f_{11}\bar{g}'_{01} + \bar{f}_{11}g'_{01} - g_{01}\bar{f}'_{11} - \bar{g}_{01}f'_{11} - g_{11}\bar{f}'_{01} - \bar{g}_{11}f'_{01}, \tag{7.4}$$

are necessary besides those given in §5. Our object is to determine the steady inflow at the far edge of the outer boundary layer: to this end, we must integrate (7.4). (It follows from (5.14) that $f_{10}+\bar{f}_{10}\to 0$ as $\eta\to\infty$.) This is not difficult. The boundary conditions are:

$$\left.\begin{aligned} &f_{01}(0) = 0, \quad f'_{01}(0) = -\tfrac{1}{2}i, \\ &f_{11}(0) = \frac{1+i}{4}, \\ &f_{10}(0) = 0, \quad f_{20}(0) = c, \end{aligned}\right\} \tag{7.5}$$

from matching to the inner solution, and thus

$$(f_{20}+\bar{f}_{20}) = i(\bar{f}_{01}f'_{11}+\bar{f}_{11}f'_{01}-f_{01}\bar{f}'_{11}-f_{11}\bar{f}'_{01})+(D+\bar{D}), \tag{7.6}$$

where

$$D = c-\tfrac{1}{8}. \tag{7.7}$$

Hence $f_{20}\to D$ on leaving the outer boundary, and this steady velocity will be preserved across the interior region, along with the azimuthal perturbation velocity of $\epsilon\mathscr{K}$, until the second disk is approached.

Ekman layer

The steady velocities are brought to rest at the second disk by an Ekman layer. The solution can be shown to be

$$\left.\begin{aligned} F_{10} &= \frac{1}{4}\left[H\exp\left(\frac{1+i}{\sqrt{2}}(z-h)\right)+J\exp\left(\frac{1-i}{\sqrt{2}}(z-h)\right)+4D\right], \\ G_{10} &= \frac{1}{4}\left[4\mathscr{K}-\frac{1-i}{\sqrt{2}}H\exp\left(\frac{1+i}{\sqrt{2}}(z-h)\right)-\frac{1+i}{\sqrt{2}}J\exp\left(\frac{1-i}{\sqrt{2}}(z-h)\right)\right], \end{aligned}\right\} \tag{7.8}$$

where h is the dimensionless distance between the two disks; the solution quoted is a limiting form for large h. H and J are unknown constants and, to satisfy the three homogeneous boundary conditions at $z = h$, it follows that

$$\sqrt{2}D-\mathscr{K} = 0. \tag{7.9}$$

Finally, combining this with (7.2) and (7.7), we deduce that

$$\mathscr{K} = \frac{15\sqrt{2}-29}{34} < 0, \tag{7.10}$$

which means the interior rotates at a slower speed than the basic rotational state.

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